Tutorial 8

In the following problems $\mathbb{F} = \mathbb{C}$ and V denotes a finite-dimensional vector space.

1. Let $T \in \mathcal{L}(V)$ be a linear operator and $U \subseteq V$ be a T-invariant subspace. Denote the minimal polynomials of T and $T|_U$ as p and p_U, respectively. Show that p_U divides p.

2. Suppose V is an inner product space and $P_U \in \mathcal{L}(V)$ is the orthogonal projection operator onto some subspace U. Assume $U \neq \{0\}$ and $U \neq V$.

   (a) What is the minimal polynomial of P_U?

   (b) What is the characteristic polynomial of P_U?

3. Let $S, T \in \mathcal{L}(V)$ be linear operators and q_S be the characteristic polynomial of S. Show that $q_S(T)$ is invertible if and only if S and T have no common eigenvalues.

4. Given a fixed $T \in \mathcal{L}(V)$ define

$$W = \operatorname{span}\{T^k \in \mathcal{L}(V) \colon k \geq 0\}$$

   Show that $\dim W = \deg p$, where p is the minimal polynomial of T.

5. Let $p(z) = \sum_{j=0}^{k} a_j z^j$ be the minimal polynomial of $T \in \mathcal{L}(V)$. Suppose T is invertible.

   (a) What can you say about $p(z)$?

   (b) What is the minimal polynomial of T^{-1}?

   (c) If V is an inner product space, what is the minimal polynomial of T^*?

6. Let $A = \begin{pmatrix} 0 & 1 \\ -4 & 4 \end{pmatrix}$. What is A^n for $n \in \mathbb{N}$? *Hint: the only eigenvalue of A is $\lambda = 2$.*

7. Let $T \in \mathcal{L}(V)$ be a linear operator and suppose $V = W_1 \oplus W_2$ for some T-invariant subspaces W_1 and W_2.

   (a) Suppose q_1 and q_2 are the characteristic polynomials of $T|_{W_1}$ and $T|_{W_2}$, respectively. Is it true that $q_1 q_2$ is the characteristic polynomial of T?

   (b) Suppose p_1 and p_2 are the minimal polynomials of $T|_{W_1}$ and $T|_{W_2}$, respectively. Is it true that $p_1 p_2$ is the minimal polynomial of T?